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Grammars for icosahedral Danzer tilings

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Abstract. We have recently interpreted 2D quasi-periodic patterns in terms of substitutions. In this paper we extend this interpretation to 3D. In particular we describe Danzer tilings in terms of word sequences of L systems.

1. Introduction

Tilings with icosahedral symmetry were obtained by Danzer in 1989 by inflation of four tetrahedra [1, 2]. Recently these tilings have been derived by projection from the root lattice D_6 [3]. On the other hand, in 1986 Socolar and Steinhardt [4] introduced a family of quasi-periodic tilings in 3D by four rhombic zonohedra. Roth [5] and Danzer *et al* [6] have independently shown the equivalence of the Danzer tilings with the tilings of Socolar and Steinhardt.

The very well known interpretation of the Fibonacci chain in terms of a substitutional sequence has been extended in [7] to 2D. In that model a bracket structure has been introduced by means of production rules of a context free grammar (see [8] for the associated automaton). Not all the words belonging to the language generated by that grammar represent pieces of the patterns. It is possible to avoid this problem by using a different type of formal grammar known as the D0L system [9]. Within the D0L-grammar approach all the words represent parts of the infinite patterns.

L systems were originated by Lindenmayer in 1968 to model biological developments in which parts of the developing organism change simultaneously. In the description of the development of a red alga he also used a bracket structure. The simplest type of L system is known as a D0L system.

A D0L system (see [10]) is a triple $G = \{\Sigma, h, \omega\}$ where Σ is an alphabet, h is an endomorphism defined on the set Σ^* of all the words over the alphabet Σ, and ω, referred to as the axiom, is an element of Σ^*. The word sequence generated by G consists of the words $h^0(\omega) = \omega, h(\omega), h^2(\omega), h^3(\omega), \ldots$ and the language of G is defined by $L(G) = \{h^i(\omega)/i \geqslant 0\}$. In what follows the endomorphism h will be defined by listing the productions for each letter. In the abbreviation D0L, 0 means that the rewriting is context independent (initially, communication between the individual cells is zero-sided in the development) and D stands for deterministic: there is just one production for each letter, i.e. the totality of all productions defines an endomorphism on Σ^*.

2. The 2D case: Penrose and triangle patterns

The triangle patterns were obtained in [11] by projection from the root lattice A_4 and they have the same basic tiles as the Robinson decomposition of the Penrose patterns [12]. In

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this section we describe the patterns via word sequences of a D0L system.

The alphabet is $\Sigma = \{a_i, b_i, c_i, d_i, (,)\}$ where $i \in \mathbb{Z}_{10}$. The elements a_i, c_i on one hand, and b_i, d_i on the other hand represent tiles with the same shape but they are distinguished by a colour: white for a_i, b_i ,black for c_i and d_i. The tile a_i is an acute isosceles triangle with (length of side)/(length of base) $= \tau = (1+\sqrt{5})/2$, and b_i represents an obtuse triangle with (length of side)/(length of base) $= 1/\tau$. The sides of b_i have the same length as the basis of a_i. We choose the same orientation for one side of the tile a_1 and the basis of b_1 in such a way that the remaining tiles are obtained by succesive rotations of $2\pi/10$ through a given vertex [7].

Every element belonging to Σ representing a tile can be used as an axiom. We take, for instance, $\omega = a_1$. The set of production rules for triangle patterns is (compare to [7])

$$\begin{aligned} &a_i \longmapsto (\Phi_T[a_i]) = (c_{i-5}b_{i+3}a_{i-2}) \qquad && b_i \longmapsto (\Phi_T[b_i]) = (c_{i+1}b_{i-1}) \\ &c_i \longmapsto (\Phi_T[c_i]) = (a_{i-5}d_{i+1}c_{i+2}) && d_i \longmapsto (\Phi_T[d_i]) = (a_{i-5}d_{i+1}) \\ &(\longmapsto (&&) \longmapsto) \end{aligned} \tag{1}$$

and for Penrose patterns

$$\begin{aligned} &a_i \longmapsto (\Phi_P[a_i]) = (a_{i+2}c_{i+1}b_{i-1}) \qquad && b_i \longmapsto (\Phi_P[b_i]) = (c_{i+1}b_{i-1}) \\ &c_i \longmapsto (\Phi_P[c_i]) = (c_{i-2}a_{i-1}d_{i+5}) && d_i \longmapsto (\Phi_P[d_i]) = (a_{i-5}d_{i+1}) \\ &(\longmapsto (&&) \longmapsto). \end{aligned} \tag{2}$$

Consider the following word derivation for the triangle patterns:

$$a_1 \longmapsto (c_6b_4a_9) \longmapsto ((a_1d_7c_8)(c_5b_3)(c_4b_2a_7)).$$

In the second word of the sequence, if two letters follow one another, the corresponding oriented triangles are glued face to face in a unique way. The word $((a_1d_7c_8)(c_5b_3)(c_4b_2a_7))$ represents a part of a triangle pattern: the acute triangle $a_1d_7c_8$, the obtuse triangle c_5b_3 and the acute triangle $c_4b_2a_7$ are glued face to face disregarding their internal composition, and again the prescription is unique.

In order to study the symmetries of the patterns we consider the Coxeter group $H_2 = \langle R_1, R_2 / (R_1R_2)^5 = R_1^2 = R_2^2 = e\rangle$. Now we define the action of H_2 on the elements of Σ representing tiles:

$$\begin{aligned} &R_1(a_i) = c_{9-i} \qquad && R_1(b_i) = d_{3-i} \qquad && R_1(c_i) = a_{9-i} \qquad && R_1(d_i) = b_{3-i} \\ &R_2(a_i) = c_{7-i} && R_2(b_i) = d_{1-i} && R_2(c_i) = a_{7-i} && R_2(d_i) = b_{1-i}. \end{aligned}$$

We have $R_k\Phi_P^n = \Phi_P^n R_k$ $(k = 1, 2)$ and also $R_k\Phi_T^n = \Phi_T^n R_k$.

In [7] we had a different commutation property for the triangle pattern case. Observe that we obtain a different derivation for the patterns with (1).

3. The oriented tiles in icosahedral Danzer tilings

The Danzer tetrahedra $T = A, B, C, K$ have the property that every plane containing a facet t^i $(t = a, b, c, k$ and $i = 1, 2, 3, 4)$ of T is parallel to one of the fifteen mirror planes of a fixed regular icosahedron. The facets t^i in figure 1 are indicated by the index $i = 1, 2, 3, 4$. In table 1 we show the dihedral angle and the lengths of the edges belonging to pairs of facets $t^i \cap t^j$ (see [2]).

If we choose a cubic coordinate system (see for instance [13, 14]) based on a set of three orthogonal 2-fold axes of the icosahedron, the Miller indices of a plane and the direction perpendicular to it are the same. The general form of the indices for the 2-fold axes is

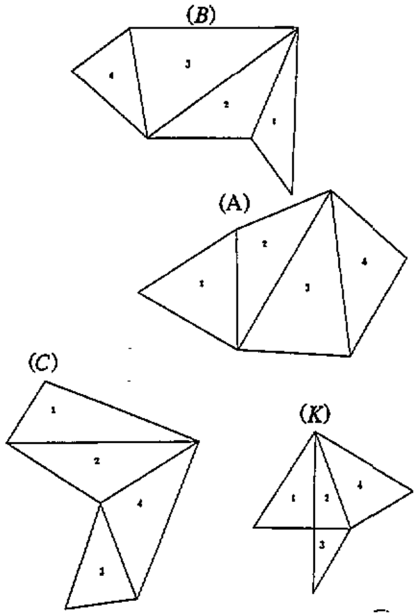

Figure 1. Outer (inner) facets of the unfolded tetrahedra A, B, C, K ($\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{K}$).

Table 1. Definition of the tetrahedra A, B, C, K.

	Edge					
Tetrahedron	$t^1 \cap t^2$	$t^2 \cap t^3$	$t^3 \cap t^1$	$t^2 \cap t^4$	$t^1 \cap t^4$	$t^3 \cap t^4$
A	$\frac{1}{2}\pi, 1$	$\frac{2}{5}\pi, \tau\rho$	$\frac{1}{5}\pi, \rho$	$\frac{1}{3}\pi, \sigma$	$\frac{3}{5}\pi, \rho$	$\frac{1}{3}\pi, \tau\sigma$
B	$\frac{1}{2}\pi, 1$	$\frac{1}{5}\pi, \tau\rho$	$\frac{1}{3}\pi, \tau\sigma$	$\frac{2}{3}2\pi, \sigma$	$\frac{3}{5}\pi, \tau^{-1}\rho$	$\frac{1}{5}\pi, \rho$
C	$\frac{1}{2}\pi, \tau$	$\frac{2}{5}\pi, \rho$	$\frac{1}{5}\pi, \tau^{-1}\rho$	$\frac{1}{5}\pi, \rho$	$\frac{1}{3}\pi, \tau\sigma$	$\frac{2}{3}\pi, \sigma$
K	$\frac{1}{2}\pi, \frac{1}{2}\tau$	$\frac{1}{2}\pi, \frac{1}{2}\tau^{-1}$	$\frac{1}{2}\pi, \frac{1}{2}$	$\frac{1}{3}\pi, \sigma$	$\frac{1}{5}\pi, \rho$	$\frac{2}{5}\pi, \tau^{-1}\rho$
		$\rho = \frac{1}{4}\sqrt{10+2\sqrt{5}}$		$\sigma = \frac{1}{2}\sqrt{3}$		

$(h + h'\tau, k + k'\tau, l + l'\tau)$ with h, h', k, k', l, l' integers. We choose the labelling for the mirror planes as

$$
\begin{array}{lll}
P_1 = (\tau, 1, 1+\tau) & P_2 = (-2\tau, 0, 0) & P_3 = (-1-\tau, \tau, 1) \\
P_4 = (0, 0, 2\tau) & P_5 = (-\tau, 1, 1+\tau) & P_6 = (-\tau, -1, 1+\tau) \\
P_7 = (\tau, -1, 1+\tau) & P_8 = (-1, -1-\tau, \tau) & P_9 = (1+\tau, \tau, 1) \\
P_{10} = (-1, 1+\tau, \tau) & P_{11} = (1, 1+\tau, \tau) & P_{12} = (-1-\tau, -\tau, 1) \\
P_{13} = (1+\tau, -\tau, 1) & P_{14} = (1, -1-\tau, \tau) & P_{15} = (0, -2\tau, 0).
\end{array}
$$

We will use eight basic tetrahedra represented by the letters $T = A, B, C, K, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{K}$. The tetrahedra T and $\widetilde{T}$ are mirror images (see figure 1) where, in this case and in what follows, we will suppose $\widetilde{\widetilde{T}} \equiv T$. In order to obtain all of the allowed oriented tiles we give an indexing to the letters. The notation $T_{\alpha,\beta}^{\gamma,\delta}$ with $\alpha, \beta, \gamma, \delta = 1, 2, \ldots, 15$ means that the tile T has its facets t^1, t^2, t^3 and t^4 lying on planes parallel to the mirror planes P_α, P_β, P_γ and P_δ respectively.

Consider the Coxeter group $H_3 = \langle R_1, R_2, R_3 / (R_1R_2)^3 = (R_2R_3)^5 = (R_3R_1)^2 = R_1^2 = R_2^2 = R_3^2 = e \rangle$. We take the reflections on the planes P_1, P_2 and P_3 as the generators of R_1, R_2 and R_3 respectively. In figure 2 we show how the mirror planes are transformed under the action of the generators of H_3. If two boxes containing P_α and P_β are joined by a line crossed by $k = 1, 2, 3$ bars then $P_\alpha = R_k[P_\beta]$. The action of the generator R_k in $T^{\gamma,\delta}_{\alpha,\beta}$ is $R_k(T^{\gamma,\delta}_{\alpha,\beta}) = \widetilde{T}^{\bar{\gamma},\bar{\delta}}_{\bar{\alpha},\bar{\beta}}$ if $R_k(P_\omega) = P_{\bar{\omega}}$ with $\omega = \alpha, \beta, \gamma, \delta$.

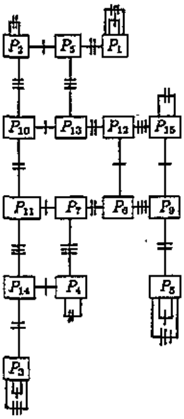

Figure 2. Transformation of the mirror planes P_α under H_3.

The allowed oriented tetrahedra and the transformations between them under the action of the generators of H_3 are shown in figures 3–6 where ordered sets of four tetrahedra are enclosed in boxes. Tetrahedra inside the same box have their facets t^1, t^2 (for A, B, C) or k^1, k^2, k^3 lying on parallel planes (we remark that the tetrahedra can be glued together to form octahedra). Lines crossed by $k = 1, 2, 3$ bars join two boxes and broken lines or broken arrows also crossed by k bars join two tetrahedra inside the same box. If the boxes $\boxed{T_1\ T_2\ T_3\ T_4}$ and $\boxed{T_1'\ T_2'\ T_3'\ T_4'}$ are joined by a line crossed by k bars then the tetrahedra with the same position $i = 1, 2, 3, 4$ inside the boxes are transformed according to $T_i = R_k[T_i']$. On the other hand, if two tetrahedra T_i and T_j inside the same box are joined by a broken line then $T_i = R_k[T_j]$ and if they are joined by a broken arrow $T_i = R_k[\widetilde{T_j}]$. If, given a diagram, we make the transformation $T \to \widetilde{T}$, we obtain another allowed diagram. If we use the notation $R_iR_j \ldots R_k \equiv R_{ij\ldots k}$ and if we read from right to left we have, for instance, $R_{132323123213232}[A^{4,9}_{1,3}] = \widetilde{A}^{4,9}_{1,3}$ (see figure 3).

4. Substitutional sequences for Danzer tilings

As in the 2D case we can interpret the tilings in terms of languages generated by grammars of type DOL. The geometric interpretation of the words is similar: if two letters follow one another the corresponding oriented tetrahedra are glued facet to facet in a unique way.

The alphabet is now

$$\Sigma = \{A^{\gamma,\delta}_{\alpha,\beta}, B^{\gamma,\delta}_{\alpha,\beta}, C^{\gamma,\delta}_{\alpha,\beta}, K^{\gamma,\delta}_{\alpha,\beta}, \widetilde{A}^{\gamma,\delta}_{\alpha,\beta}, \widetilde{B}^{\gamma,\delta}_{\alpha,\beta}, \widetilde{C}^{\gamma,\delta}_{\alpha,\beta}, \widetilde{K}^{\gamma,\delta}_{\alpha,\beta}, (,)\}.$$

Every element of the alphabet except the brackets) and (can be used as an axiom. For the production rules we choose an ordering compatible with the inflation of the tetrahedra facets. It is possible to choose such an order in all the cases except one. In order to obtain the correct inflation for that facet we must introduce a bracket in one index.

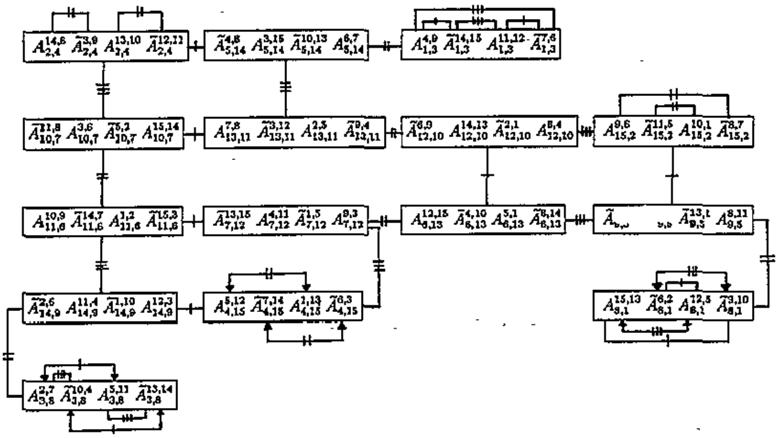

Figure 3. Transformation of the tetrahedra $A^{\gamma,\delta}_{\alpha,\beta}$ under H_3.

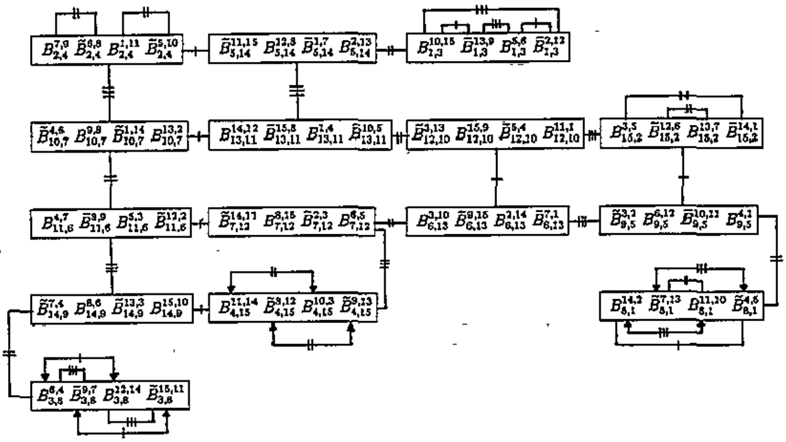

Figure 4. Transformation of the tetrahedra $B^{\gamma,\delta}_{\alpha,\beta}$ under H_3.

The set of production rules is

$$\begin{aligned} &K^{\gamma,\delta}_{\alpha,\beta} \longmapsto (\Phi_{\mathrm{D}}[K^{\gamma,\delta}_{\alpha,\beta}]) && C^{\gamma,\delta}_{\alpha,\beta} \longmapsto (\Phi_{\mathrm{D}}[C^{\gamma,\delta}_{\alpha,\beta}]) \\ &B^{\gamma,\delta}_{\alpha,\beta} \longmapsto (\Phi_{\mathrm{D}}[B^{\gamma,\delta}_{\alpha,\beta}]) && A^{\gamma,\delta}_{\alpha,\beta} \longmapsto (\Phi_{\mathrm{D}}[A^{\gamma,\delta}_{\alpha,\beta}]) = (\Phi_{\mathrm{D}}[\widetilde{B}^{\gamma,\delta}_{(\alpha),\beta}]Z^{\gamma,\delta}_{(\alpha),\beta}) \\ &(\longmapsto (&&) \longmapsto) \end{aligned} \quad (3)$$

with

$$\Phi_{\mathrm{D}}[K^{\gamma,\delta}_{\alpha,\beta}] = K^{\beta,\epsilon}_{\gamma,\alpha} B^{\delta,\epsilon}_{\beta,\alpha} \qquad \Phi_{\mathrm{D}}[C^{\gamma,\delta}_{\alpha,\beta}] = \widetilde{K}^{\epsilon_1,\delta}_{\beta,\alpha} K^{\epsilon_1,\epsilon_2}_{\beta,\alpha} C^{\epsilon_2,\alpha}_{\delta,\epsilon_3} \widetilde{C}^{\beta,\epsilon_4}_{\delta,\epsilon_3} A^{\gamma,\epsilon_4}_{\alpha,\beta}$$

$$\Phi_{\mathrm{D}}[B^{\gamma,\delta}_{\alpha,\beta}] = K^{\epsilon_2,\alpha}_{\epsilon_1,\gamma} \widetilde{K}^{\epsilon_2,\epsilon_3}_{\epsilon_1,\gamma} \widetilde{B}^{\delta,\epsilon_3}_{\epsilon_2,\gamma} B^{\beta,\epsilon_4}_{\epsilon_2,\gamma} K^{\epsilon_2,\epsilon_4}_{\epsilon_1,\gamma} \widetilde{K}^{\epsilon_2,\epsilon_5}_{\epsilon_1,\gamma} \widetilde{C}^{\epsilon_5,\gamma}_{\alpha,\beta} \qquad Z^{\epsilon,\delta}_{\alpha,\beta} = \widetilde{C}^{\epsilon_1,\beta}_{\delta,\epsilon} \widetilde{K}^{\alpha,\epsilon_1}_{\epsilon_5,\beta} K^{\alpha,\epsilon_6}_{\epsilon_5,\beta} B^{\epsilon,\epsilon_6}_{\alpha,\beta}. \quad (4)$$

The rules for $\widetilde{T}$ are obtained by making the transformation $T \to \widetilde{T}$.

The common facet θ of two tiles corresponding to two consecutive letters is $K^{*,\theta}_{*,*} B^{*,\theta}_{*,*}$, $K^{*,\theta}_{*,*} C^{\theta,*}_{*,*}$, $C^{*,*}_{*,\theta} \widetilde{C}^{*,*}_{*,\theta}$, $\widetilde{C}^{*,\theta}_{*,*} A^{*,\theta}_{*,*}$, $\widetilde{B}^{*,*}_{\theta,*} B^{*,*}_{\theta,*}$. For the coupling $\widetilde{K}K$ we have $\widetilde{K}^{\theta,*}_{*,*} K^{\theta,*}_{*,*}$ in

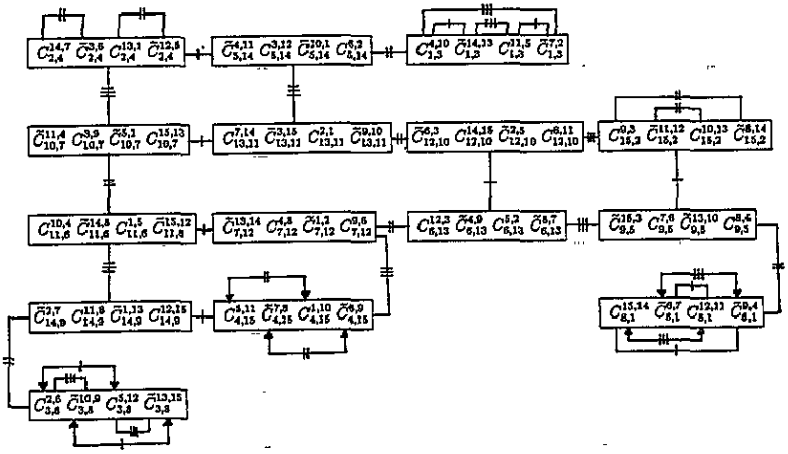

Figure 5. Transformation of the tetrahedra $C^{\gamma,\delta}_{\alpha,\beta}$ under H_3.

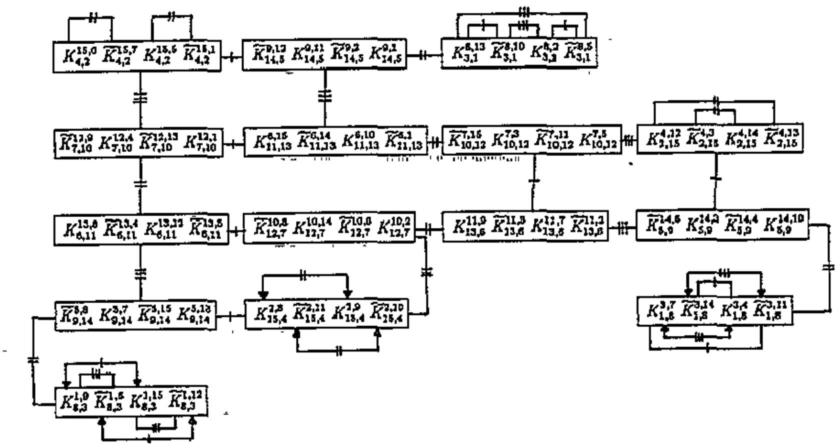

Figure 6. Transformation of the tetrahedra $K^{\gamma,\delta}_{\alpha,\beta}$ under H_3.

$\Phi_D[C^{\gamma,\delta}_{\alpha,\beta}]$ and $\widetilde{K}^{*,*}_{\theta,*}K^{*,*}_{\theta,*}$ in $\Phi_D[A^{\gamma,\delta}_{\alpha,\beta}]$ and $\Phi_D[B^{\gamma,\delta}_{\alpha,\beta}]$. We also have the same common facets if we permute the letters or we transform $T \to \widetilde{T}$.

From the set of production rules we can also see how the facets are inflated. For instance, if we want to know how the facet b^3 is inflated we must look for facets lying on planes parallel to P_γ. We obtain $b^3 \longmapsto k^2\widetilde{k}^2\widetilde{b}^2b^2k^2\widetilde{k}^2\widetilde{c}^4$. Again, if two letters follow one another the corresponding triangles are glued edge to edge. The brackets are necessary in order to obtain the correct inflation for the facet a_1: in this case $a_1 \longmapsto (b^3)(\widetilde{k}^3k^3b^1)$. If we apply the production rules to one letter several times, we must be careful if we want to see how the facets are inflated and look for facets which do not correspond to common facets of two tiles represented by consecutive letters (see example 2 below).

We can derive the tilings by following one of the paths A or B.

(A) We give the production rules for a fixed set of tiles. The remaining production rules are obtained by applying $\Phi_D R_k = R_k \Phi_D$ where R_k are the generators of H_3.

(B) We use the production rules given in (3) and (4) and the diagrams from figures 3–6.

If we follow path A we must first give the production rules for a fixed set of tiles:

$$\Phi_{\mathrm{D}}[K_{3,1}^{8,13}] = K_{8,3}^{1,9}B_{1,3}^{13,9}$$
$$\Phi_{\mathrm{D}}[C_{1,3}^{4,10}] = \widetilde{K}_{3,1}^{8,10}K_{3,1}^{8,5}C_{10,7}^{5,1}\widetilde{C}_{10,7}^{3,9}A_{1,3}^{4,9}$$
$$\Phi_{\mathrm{D}}[B_{1,3}^{10,15}] = K_{7,10}^{12,1}\widetilde{K}_{7,10}^{12,9}\widetilde{B}_{12,10}^{15,9}B_{12,10}^{3,13}K_{7,10}^{12,13}\widetilde{K}_{7,10}^{12,4}\widetilde{C}_{1,3}^{4,10}$$
$$\Phi_{\mathrm{D}}[A_{1,3}^{4,9}] = \widetilde{K}_{15,4}^{2,9}K_{15,4}^{2,11}B_{2,4}^{1,11}\widetilde{B}_{2,4}^{5,10}\widetilde{K}_{15,4}^{2,10}K_{15,4}^{2,8}C_{9,5}^{8,4}\widetilde{C}_{9,5}^{15,3}\widetilde{K}_{8,3}^{1,15}K_{8,3}^{1,6}B_{1,3}^{5,6}.$$

To illustrate the procedure we give two examples.

Example 1:

$$\begin{aligned}K_{3,1}^{8,13} \longmapsto (K_{8,3}^{1,9}B_{1,3}^{13,9}) &\longmapsto ((\Phi_{\mathrm{D}}[K_{8,3}^{1,9}])(\Phi_{\mathrm{D}}[B_{1,3}^{13,9}]))\\ &= ((\Phi_{\mathrm{D}}[R_{232312}[K_{3,1}^{8,13}]])(\Phi_{\mathrm{D}}[R_{32312132132312}[B_{1,3}^{10,15}]]))\\ &= ((R_{232312}[\Phi_{\mathrm{D}}[K_{3,1}^{8,13}]])(R_{32312132132312}[\Phi_{\mathrm{D}}[B_{1,3}^{10,15}]]))\\ &= ((K_{1,8}^{3,7}B_{3,8}^{9,7})(K_{11,13}^{6,1}\widetilde{K}_{11,13}^{6,15}\widetilde{B}_{6,13}^{9,15}B_{6,13}^{3,10}K_{11,13}^{6,10}\widetilde{K}_{11,13}^{6,14}\widetilde{C}_{1,3}^{14,13})).\end{aligned}$$

We can now check that the facets are inflated in the right way:

$$k^1 \longmapsto ((k^3b^1)(b^3\widetilde{c}^2)) \qquad k^2 \longmapsto ((k^1)(k^4\widetilde{c}^1))$$
$$k^3 \longmapsto ((k^2b^2)) \qquad k^4 \longmapsto ((k^2\widetilde{k}^2\widetilde{b}^2k^2\widetilde{k}^2\widetilde{c}^4)).$$

Example 2:

$$\begin{aligned}C_{1,3}^{4,10} \longmapsto\ & \widetilde{K}_{3,1}^{8,10}K_{3,1}^{8,5}C_{10,7}^{5,1}\widetilde{C}_{10,7}^{3,9}A_{1,3}^{4,9}\\ \longmapsto\ & ((\Phi_{\mathrm{D}}[R_1[K_{3,1}^{8,13}]])(\Phi_{\mathrm{D}}[R_3[\widetilde{K}_{3,1}^{8,13}]])(\Phi_{\mathrm{D}}[R_{31231}[\widetilde{C}_{1,3}^{4,10}]])\\ & \times(\Phi_{\mathrm{D}}[R_{3121}[\widetilde{C}_{1,3}^{4,10}]])(\Phi_{\mathrm{D}}[A_{1,3}^{4,9}]))\\ =\ & ((\widetilde{K}_{8,3}^{1,15}\widetilde{B}_{1,3}^{10,15})(K_{8,3}^{1,6}B_{1,3}^{5,6})(\widetilde{K}_{7,10}^{12,1}K_{7,10}^{12,4}C_{1,3}^{4,10}\widetilde{C}_{1,3}^{7,2}A_{10,7}^{5,2})\\ & \times(K_{7,10}^{12,9}\widetilde{K}_{7,10}^{12,13}\widetilde{C}_{9,5}^{13,10}C_{9,5}^{7,6}\widetilde{A}_{10,7}^{3,6})\\ & \times(\widetilde{K}_{15,4}^{2,9}K_{15,4}^{2,11}B_{2,4}^{1,11}\widetilde{B}_{2,4}^{5,10}\widetilde{K}_{15,4}^{2,10}K_{15,4}^{2,8}C_{9,5}^{8,4}\widetilde{C}_{9,5}^{15,3}\widetilde{K}_{8,3}^{1,15}K_{8,3}^{1,6}B_{1,3}^{5,6})).\end{aligned}$$

The facet c^2 lying on a plane parallel with P_3 can be found in the pair $C_{1,3}^{4,10}\widetilde{C}_{1,3}^{7,2}$, but it corresponds to a inner coupling between the facets. We can obtain the correct inflation for the facets if we look only into the facets that do not correspond to inner couplings of the tetrahedra:

$$c^1 \longmapsto ((\widetilde{k}^3\widetilde{b}^1)(k^3b^1)(\widetilde{k}^4c^1\widetilde{c}^1)(b^3\widetilde{k}^3k^3b^1)) \qquad c^2 \longmapsto ((\widetilde{k}^2\widetilde{b}^2)(k^2b^2)(\widetilde{a}^3)(\widetilde{c}^4\widetilde{k}^2k^2b^2))$$
$$c^3 \longmapsto ((\widetilde{k}^2k^2b^2\widetilde{b}^2\widetilde{k}^2k^2c^4)) \qquad c^4 \longmapsto ((\widetilde{b}^3)(\widetilde{k}^2k^2c^4a^1)(k^2\widetilde{k}^2\widetilde{c}^4\widetilde{a}^1)).$$

In order to derive the tilings following path B let us see how the sets of indices ϵ_i can be obtained. The diagrams in figures 3–6 show that for fixed values of α, β, γ and δ the indices x and y in $T_{\alpha,\beta}^{\gamma,x}$, $T_{\alpha,\beta}^{x,\delta}$, $T_{x,y}^{\gamma,\delta}$ ($T = A, B, C$) and in $K_{x,\beta}^{y,\delta}$ are unique. This property also holds for the tiles of type $\widetilde{T}$. We can obtain the indices of type ϵ in the following way (the fact that the index ϵ is obtained by looking for $T_{\alpha,\beta}^{\epsilon,\gamma}$ in the diagrams is indicated by $T_{\alpha,\beta}^{\epsilon,\gamma} \Rightarrow \epsilon$):

(i) $[\epsilon]$ in $\Phi_{\mathrm{D}}[K_{\alpha,\beta}^{\delta,\gamma}]$:

$$B_{\beta,\alpha}^{\delta,\epsilon} \Rightarrow \epsilon$$

(ii) $[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]$ in $\Phi_{\mathrm{D}}[C_{\alpha,\beta}^{\delta,\gamma}]$:

$$\widetilde{K}_{\beta,\alpha}^{\epsilon_1,\delta} \Rightarrow \epsilon_1 \qquad A_{\alpha,\beta}^{\gamma,\epsilon_4} \Rightarrow \epsilon_4 \qquad \widetilde{C}_{\delta,\epsilon_3}^{\beta,\epsilon_4} \Rightarrow \epsilon_3 \qquad C_{\delta,\epsilon_3}^{\epsilon_2,\alpha} \Rightarrow \epsilon_2$$

(iii) $[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5]$ in $\Phi_D[B^{\gamma,\delta}_{\alpha,\beta}]$:

$$K^{\epsilon_2,\alpha}_{\epsilon_1,\gamma} \Rightarrow \epsilon_1, \epsilon_2 \qquad \widetilde{B}^{\delta,\epsilon_3}_{\epsilon_2,\gamma} \Rightarrow \epsilon_3 \qquad B^{\beta,\epsilon_4}_{\epsilon_2,\gamma} \Rightarrow \epsilon_4 \qquad \widetilde{C}^{\epsilon_5,\gamma}_{\alpha,\beta} \Rightarrow \epsilon_5$$

(iv) $[\epsilon, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6]$ in $\Phi_D[A^{\gamma,\delta}_{\alpha,\beta}]$:

$$\widetilde{K}^{\epsilon_2,\alpha}_{\epsilon_1,\gamma} \Rightarrow \epsilon_1, \epsilon_2 \qquad B^{\delta,\epsilon_3}_{\epsilon_2,\gamma} \Rightarrow \epsilon_3 \qquad \widetilde{C}^{\epsilon_1,\beta}_{\delta,\epsilon} \Rightarrow \epsilon$$
$$\widetilde{B}^{\beta,\epsilon_4}_{\epsilon_2,\gamma} \Rightarrow \epsilon_4 \qquad \widetilde{K}^{\alpha,\epsilon_1}_{\epsilon_5,\beta} \Rightarrow \epsilon_5 \qquad B^{\epsilon,\epsilon_6}_{\alpha,\beta} \Rightarrow \epsilon_6.$$

Observe that one has to maintain the order because sometimes the indices ϵ_i previously obtained must be known.

5. Conclusion

We have given an interpretation of Danzer tilings in terms of substitutional sequences. The words representing the tilings also give information about the way the facets of the tiles are inflated.

The notions of translations, point group and space groups of standard crystallography have been generalized in [8, 15–18], in order to construct a non-commutative crystallography. These studies are based in $\mathrm{Aut}(F_n)$ (group of automorphisms of F_n, the free group with n generators). In that scheme one can find the elements of a non-commutative crystallography in which the commutativity of the translation group is broken and the point symmetry is preserved. The relationship between the model presented in this work and the non-commutative crystallography of $\mathrm{Aut}(F_n)$ should be explored.

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